# A NOVEL METHOD FOR ESTIMATION OF MODEL PARAMETERS FOR NOISE CORRUPTED SIGNALS

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***Abstract***

*Often measured data are produced by an integration process over many layers, whereas the parameter values in every layer are of interest. The data are corrupted by noise. The estimation of the model parameters, if only the summed measured data and their relationships to the parameters (forward model) are known, is an inverse problem. The paper shows how coupled integrals near an a priori known situation can be summed up by an operator representation. It is shown a method to calculate the probability density function for model parameter estimation. The obtained expectation value with the algebraic dependence of the parameters is related to the measured value and solves the inverse problem.*

## ***Representation of the inverse problem by operators in the original non-discrete form***

Very often the measurements and their connections by coupled systems of summation with the internal model of a physical or technical system are all what is known about the interesting internal system. Given are mostly only complicated non-linear relationships between the measurement values and the internal parameters. Moreover, the measured values are coupled with more inner parameters over integrals such as a convolution, and the measured values are corrupted by noise. This is the situation for the most digital signal processing models, where the internal parameters are a higher-dimensional vector and the measure values are only a scalar. The system is underestimated because some model parameters are leaving by an intrinsic lack of data or there are experimental uncertainties. An uncertainty of knowledge is also possible, because the approximation is to simple for the more complex reality. Mathematically such a measure problem is an ill posed problem i.e. if a “small” perturbation of data there corresponds to an arbitrarily “large” perturbation of the solution. Examples for this are to find in the radar technique, seismology, atmospheric radiation, and many geophysical problems. The inverse problems are mostly solved by linear programming, least-squares and maximum likelihood methods [1]. Here a novel method is presented based on an operator representation for summing up an infinite sum of coupled integrals. The physical system may be completely described by a non-linear stochastic differential equation with a set Θ of model parameters. The parameters may not be directly measurable and the measurements are corrupted by noise. Some observable parameters can be operationally defined whose actual values hopefully depend on the values of the model parameters.

In most cases the relationships are given by the non-linear function F with the noise ξ. The measured value z is assumed to be dependent of the heights of the layer y and the interesting set Θ of system parameters by the form:

$$z(y) = \; z_0 \int_0^y F(\Theta, z(y'), \, x(y') \, dy'$$

or in differential form

$$\frac{\partial}{\partial y}\left(z(y)-z_r\right)=F(\Theta, z(y), x(y)) \qquad (1)$$

For this equation for the measurements mostly a nearly known situation exists, for example the standard-atmosphere or the undisturbed seismogram. This solution $z_r$ is used as a reference where the digital signal procession is beginning and than followed a development about this solution. The function z will be represented related to the standard solution in the differential form. The function is also developed by powers of a parameter ε . This is only an auxiliary parameter, which is set later $e=1$. With

$$z(y)-z_r \equiv H(z_r, y)=eH_1(z_r, y)+e^2 H_2(z_r, y)+\cdots \quad (2)$$

is obtained

$$\frac{\partial}{\partial y}(z(y)-z_r)=e\frac{\partial}{\partial y}H_1(z_r,y)+e^2\frac{\partial}{\partial y}H_2(z_r,y)+\cdots$$
$$=\frac{\partial}{\partial y}H(z_r, y)$$

The Taylor series of eq. 1 gives

$$\frac{\partial}{\partial y}z(y)=e\,F(z_r,y,x(y))+e\,(z-z_r)\frac{\partial F(z,y,x(y))}{\partial z}\Bigg|_{z=z_r}$$
$$+e^2\frac{(z-z_r)^2}{2}\frac{\partial F(z_r,y,x(y))}{\partial z}\Bigg|_{z=z_r}+\cdots$$

The last three upper equations with the terms of the same power of ε give the system of equations

$$H_1(z_r,y)=\int_0^y F(z_r,y',x(y'))\,dy' \ ,$$

$$H_2(z_r,y)=\int_0^y dy' F(z_r,y',x(y'))\,H_1(z_r,y')\,dy=$$
$$\int_0^y dy_1 F(z_r,y_1,x(y_1))\int_0^{y_1} F(z_r,y_2,x(y_2))\,dy_2$$

$$H_3(z_r,y)=\int_0^y dy_1 F(z_r,y_1,x(y_1))\int_0^{y_1} dy_2 F(z_r,y_2,x(y_2))$$
$$\int_0^{y_2} F(z_r,y_3,x(y_3))\,dy_3 \ , \qquad \cdots$$

The parameter set Θ is omitted for a shorter writing. Using the **T**-product [2] of the quantum field theory for the ordering in the form

$$\mathbf{T}\{F(y_1)F(y_2)\}=\begin{cases} F(y_1)F(y_2) & \text{for} \quad y_2\ge y_1 \\ F(y_2)F(y_1) & \text{for} \quad y_1>y_2 \end{cases}$$

the integral terms can be summarised.
Because the integral over the upper and the lower region is the same, for $H_2(z_r, y)$ valid

$$H_2(z_r,y)=\frac{1}{2!}\int_0^y dy_1\int_0^y dy_2\ \mathbf{T}\{F(z_r,y_2,x(y_2))\,F(z_r,y_1,x(y_1))\}$$

Analogue [3] the higher components are built and for (2) with $e=1$

$$z(y)-z_r=\mathbf{T}\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^y dy_1\cdots\int_0^y dy_n\,F(z_r,y_1)\cdots F(z_r,y_n)$$

is obtained.

***Calculation of the probability density function by shifting of operators***

The solution of the inverse problem is disturbed by noise. This effect is reduced if the measured value is replaced by the estimated value. For this is used the characteristic function

$$\left\langle e^{iu(z-z_r)}\right\rangle=1+\mathbf{T}\sum_{n=1}^{\infty}\frac{(iu)^n}{n!}\int_0^y dy_1\cdots$$
$$\int_0^y dy_n\left\langle F(z_r,y_1\,x(y_1))\cdots F(z_r,y_n,x(y_n))\right\rangle$$

where $\langle\ \rangle$ is the mean value over a statistical ensemble. The backward transformation gives the probability density function

$$p(z|z_r)=\frac{1}{2p}\int e^{-iu(z-z_r)}\left\langle e^{iu(z-z_r)}\right\rangle du=$$

$$[1+\mathbf{T}\sum_{n=1}^{\infty}\frac{1}{n!}\left(-\frac{\partial}{\partial z}\right)^n\int_0^y dy_1\cdots \tag{3}$$

$$\int_0^y dy_n\left\langle F(z,y_1x(y_1))\cdots F(z,y_n,x(y_n))\right\rangle]\,d(z-z_r)$$

This can be written more compactly as

$$p(z|z_r)=\mathbf{T}\,\exp\left\{-\frac{\partial}{\partial z}\int_0^y F(z_r,y',x(y'))\,dy'\right\}d(z-z_r)$$

By this representation the interconnection of the different contributions are represented as an operator equation. This operator equation can be solved in most cases by shifting the operator to the right side and than to omit the term with the operator. Here are given some examples:

$$e^{cz\frac{\partial}{\partial z}}z=ze^{z+cz\frac{\partial}{\partial z}}=ze^{c}\ ,$$

$$e^{c\frac{\partial}{\partial z}+bz}=e^{cb+bz+c\frac{\partial}{\partial z}}\ ,$$

$$e^{cz\frac{\partial}{\partial z}+b\frac{\partial^2}{\partial z^2}}=e^{cz\frac{\partial}{\partial z}+b\frac{\partial^2}{\partial z^2}}e^{-b\frac{\partial^2}{\partial z^2}+\frac{b}{2c}(e^{2c}-1)\frac{\partial^2}{\partial z^2}}\ ,$$

$$e^{c\frac{\partial}{\partial z}}e^{\cos(z+b)}=e^{\cos(z+b+c)}\ ,$$

$$\frac{\partial^{2n}}{\partial z^{2n}}e^{-cz^2}=c^n e^{-cz^2}(-1)^n\frac{(2n)!}{(n!)^2}\,{}_1F_1(-n,\tfrac{1}{2};cz^2)\ .$$

Where ${}_1F_1(-n,\ 0.5,\ cz^2)$ is the hypergeometric confluent function.

On the other hand, different sums in the exponential functions can be written for better calculation in a compact form such as

$$e^{a\cos(z+b)+c\cos(z+d)}=$$

$$\exp\left\{\sqrt{(A)^2+(B)^2}\cos\left(z+arctg\left(\frac{B}{C}\right)\right)\right\}$$

where $A=a\cos(b)+c\cos(d)$ and $B=a\sin(b)+c\sin(d)$.

The upper operator equations are also applied on the integrals of (3) in the form

$$e^{\int_0^y c(y')\frac{\partial}{\partial z}dy'}F(z,y)e^{-\int_0^y c(y')\frac{\partial}{\partial z}dy'}\mathbf{T}e^{\int_0^y c(y')\frac{\partial}{\partial z}dy'}= \tag{4}$$

$$F\left(z+\int_0^y c(y')\,dy',y\right)\mathbf{T}e^{\int_0^y c(y')\frac{\partial}{\partial z}dy'}$$

That means in all terms of F(z) is z to replace by $z+\int_0^y c(y')\,dy'$. It can be shown similarly, that after the shifting of the operator $\mathbf{T}\exp\int_0^y c(y')z\frac{\partial}{\partial z}dy'$ the value z is to be replaced by $z\exp\int_0^y c(y')dy'$. If the operator $\mathbf{T}\exp\int_0^y c(y')\frac{\partial^2}{\partial z^2}dy'$ is shifted to the right side, than the value z of the function F (z) is to be replaced by $z+2\int_0^y c(y')\frac{\partial}{\partial z}dy'$.

In such a way the summation of different integrals is transformed in a solving of operator equation where the differential operators are shifted to right. If all differential operators are on the right side, then the operators work only on the function $d(z-z_r)$. If this δ-function is represented by the integral

$$\int ds\, e^{2ik(z-z_r)}$$

then the shifting of the differential operator $\partial/\partial z$ on the right side has the effect that in $\exp(ikz)$ all $ik$ are replaced by $ik+\partial/\partial z$. For calculation of the probability density function of z only known integrals obtained from (5) are to solve. The estimation value of z as a function of the interesting parameters Θ is than calculated by the integration with the probability function

$$\hat{z}=\int z\, p(z,\Theta|z_r)\,dz \qquad (4)$$

This expectation value $\hat{z}$ is given as a function about the a priori solution $z_r$ of the interesting set of parameters Θ and is the optimal estimation for the measured value. For the calculation of the expectation value in non-linear case can also applied the matringale method [4].

## Representation of the discrete inverse problem for model parameter estimation

Because the solution (4) result from an infinite summation of terms, the deviation from the beginning state can be big. This infinite number of terms is also summed up if instead of an integral a sum is used. This sum can than be written as a system of coupled equation.
The measured values are obtained in a discrete form. For the equation (4) can than be written

$$\hat{z}=\sum_{i=1}^{n} z_i\, p(z_i,\Theta_i|z_{r_i})$$

where the summation is going over different parts (differnt layers, particles, spectra etc.) where every part has his own “fingerprints” (shape of curve, density distribution etc.), which is a priori known by the forward model. This gives the possibility to find an optimised approximation for $\hat{z}$ with and so to obtain the best adaptation for the interesting parameter set Θ. Because many non-linear function are damped, the form is $\exp(-cz^2)$ and gives $_1F_1$ functions which can be represented as an finite row, because they contains terms of the form (-n)*(-n+1)*(-n+2) ... . In this case all term higher then the (n-1)$^{th}$ are zero and we get a finite set of equation for the estimation of terms of z. This coupled equations can be used for calculation of the interesting parameters Θ.
If trigonometrically functions are present, than by the operator representation after the integration the same function with other parameters can be produced. In such a way algorithms for calculation step by step can be developed. In all applied examples it was possible to find iterative algorithms of the same form for calculation of the parameters Θ .
The benefit of this method is, that the infinite summation by the operator technique in combination with the representation of integration of non-linear function by an operator representation allows to develop the result near an a priori known solution without the problems in Taylor series. The problems by noise can reduced by previous non-linear filtering

### *References*

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