Modelling the dynamics of superfluid neutron stars

Nils Andersson

School of Mathematics, Univ. Southampton, UK
As \( T \rightarrow 0 \) matter either freezes to a solid or becomes superfluid.

\((\text{He}^3, \text{Bose-Einstein condensates, fermion Cooper pairing etc.)}\)

Mature neutron stars are thought to contain: superfluid neutrons, superconducting protons, superfluid hyperons and possibly also colour superconducting quarks.

Want to understand the dynamics of these phases.

- “glitches”
  What is the mechanism?
- precession
  What about vortex pinning?
- oscillations
  Imprint for asteroseismology?
- gravitational waves
  Does viscosity stabilise r-modes?
  Is mutual friction the key damping agent?
  Hyperon bulk viscosity?
- two-stream instability
  Can it operate in neutron stars?
Below the neutron superfluid transition temperature the dominant contribution to the shear viscosity is made by the scattering of relativistic electrons.

Against intuition, this leads to a **stronger** shear viscosity.

Above proton transition temperature:

\[
\eta_{ep} \approx 1.8 \times 10^{18} \left( \frac{x_p}{0.01} \right)^{13/6} \rho_{15}^{13/6} T_{8}^{-2} \text{ g/cm s}
\]

When protons are superconducting:

\[
\eta_{ee} \approx 4.4 \times 10^{19} \left( \frac{x_p}{0.01} \right)^{3/2} \rho_{15}^{3/2} T_{8}^{-2} \text{ g/cm s}
\]

The protons play the key role. Individual scattering processes add like “parallel resistors”;

\[
\tau = \left[ \frac{1}{\tau_{ee}} + \frac{1}{\tau_{ep}} \right]^{-1}
\]

where \( \tau_{ee} >> \tau_{ep} \)

The most important contribution comes from the most frequent scattering process.

**Need:** Superfluid suppression factors for finite temperature (cf. cooling)
Bulk viscosity due to non-leptonic interactions may suppress the gravitational-wave driven r-mode instability. In particular, the hyperon bulk viscosity is relevant at low temperatures (the coefficient scales as $T^{-2}$ rather than $T^6$).

But, in order not to be in conflict with cooling data, the hyperons must be superfluid.

Then the nuclear reactions that lead to the bulk viscosity are suppressed...

... and the effect on the instability may not be so great after all.

Most recent estimates suggest that r-modes may be unstable in LMXBs, and could possibly provide persistent gravitational-wave signal.

**Need:** Better hyperon gap estimates and multifluid models for $n$, $p$, $\Lambda$, $\Sigma^-$ etcetera.
The equations describing a two-fluid model for superfluid neutron stars can be derived from an energy functional $E(n_n, n_p, w^2)$ where $w^{yx}_i \equiv v^Y_i - v^x_i$ ($x, y$ are constituent indices);

$$dE = \sum_{x=n,p} \mu_x dn_x + \alpha dw^2 \rightarrow \tilde{\mu}_x = \frac{1}{m_B} \frac{\partial E}{\partial n_x} \text{ and } \alpha = \frac{\partial E}{\partial w^2}$$

In the conservative case this leads to

$$\partial_t n_x + \nabla_i (n_x v^i_x) = 0$$

and

$$n_x (\partial_t + \mathcal{L}_{v_x}) p^x_i + n_x \nabla_i \left( \mu_x - \frac{1}{2} m v^2_x \right) = 0$$

The momentum $p^x_i = m (v^x_i + \varepsilon_x w^{yx}_i)$, where $\varepsilon_x = 2\alpha/m_B n_x$, encodes the entrainment effect.

Note: The entrainment can be represented by an effective mass $m^*_p$.

Need: Entainment parameter at finite temperature (EoS “out of equilibrium”)
The presence of vortices leads to mutual friction between interpenetrating superfluids (neutrons and protons).

- superfluid neutron momentum vorticity is quantised \( \kappa^i = \epsilon^{ijk} \nabla_j p^n_k \)
- induces flow in part of the protons because of entrainment
- magnetic fields form on the vortices

\[ B \approx 2 \times 10^{14} \text{G} \left( \frac{x_p}{0.05} \right) \left( \frac{\rho}{10^{14} \text{g/cm}^3} \right) \left| \frac{m_p - m^*_p}{m^*_p} \right| \]

- electrons scatter dissipatively off these magnetic fields

Balancing the Magnus force to force due to electrons scattering off of the vortex

\[ f^M_i = \rho_n \epsilon^{ijk} \kappa_j w^n_{kL} = \mathcal{R}(v^p_i - v^L_i) = f^e_i \]

one can infer that the mutual friction force acting on the neutrons is

\[ f^i_{mf} = B \rho_n n_v \epsilon^{ijk} \kappa_j \epsilon_{klm} \kappa^l w^m_{pn} + B' \rho_n n_v \epsilon^{ijk} \kappa^l w^l_{pn} \]

\[ \text{projects} \perp \kappa^i \]

\[ \text{acts} \perp w^p_{pn} \]

**Need:** Understanding of possible vortex “clusters".
The dimensionless coefficients are $B' = B^2$ and
\[
B = \frac{\mathcal{R}}{\rho_n \kappa} \approx 4 \times 10^{-4} \left( \frac{m_p - m_p^*}{m_p} \right)^2 \left( \frac{m_p}{m_p^*} \right)^{1/2} \left( \frac{x_p}{0.05} \right)^{7/6} \left( \frac{\rho}{10^{14} \text{g/cm}^3} \right)^{1/6}
\]

Working out dynamical coupling timescale we find
\[
\begin{align*}
n_n \partial_t p^n_i + \ldots &= f_i^{mf} \\
n_p \partial_t p^p_i + \ldots &= -f_i^{mf}
\end{align*}
\]

That is, the timescale on which the two fluids are (locally) coupled can be estimated as
\[
\tau_d \approx \frac{m_p^* x_p}{m_p B \kappa n_v} \approx 10 P(s) \left( \frac{m_p^*}{m_p - m_p^*} \right)^2 \left( \frac{x_p}{0.05} \right)^{-1/6} \left( \frac{\rho}{10^{14} \text{g/cm}^3} \right)^{-1/6}
\]

This is about 1 order of magnitude faster than the estimate by Alpar and Sauls (1988). Usually taken as evidence that glitches must originate in the crust superfluid.

**Need:** Understanding of “pinning” and role of turbulence (vortex tangles).
Current to-do list

- develop a flux-conservative formalism for multi-fluid systems, including dissipation and entrainment.

Note: The simplest model for neutron stars, starting with four fluids and reducing to two degrees of freedom, requires 19 dissipation coefficients.

- consider relativistic models for superfluids coexisting with an elastic crust. This should allow us to build “quantitative” models for glitch relaxation.

- worry about $T \neq 0$ (probably always relevant!). How many “fluids” do you need? How account for excitations?

- think about superfluid “turbulence”. Is it relevant for neutron stars? If so, how does it manifest itself?

- learn more about “exotica”. How does colour superconductivity work?