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Covariant Compton Scattering Kernel in General Relativistic Radiative Transfer

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ABSTRACT

To formulate general relativistic radiative transfer with Compton scattering requires a covariant scattering kernel. An explicit closed-form expression for such a kernel is unfortunately unavailable. Previous attempts to derive it were only partially successful. Here we present a closed-form expression of the scattering kernel in terms of hypergeometric functions. The derivations are shown explicitly. With such a close-form scattering kernel, the computation task to solve the general relativistic radiative transfer equation with explicit treatment of Compton scattering is greatly reduced.

Key words: radiative transfer – scattering – relativity.

1 INTRODUCTION

In Newtonian space-time the radiative transfer equation in a medium reads

$$\left(\frac{1}{c}\frac{\partial}{\partial t}+\hat{\Omega}\cdot\nabla\right)I_{\nu}(\hat{\Omega})=j_{\nu}(\hat{\Omega})-\kappa_{\nu}I_{\nu}(\hat{\Omega})+\iint d\Omega\;d\nu'\;\sigma(\nu,\hat{\Omega};\nu',\hat{\Omega}')I_{\nu'}(\hat{\Omega}') \tag{1}$$

(see Mihalas & Mihalas 1984; Peraiah 2001) where $I_{\nu}(\hat{\Omega})$ is the intensity of the radiation at a frequency ν propagating in the $\hat{\Omega}$-direction, j_{ν} and κ_{ν} are the emission and absorption coefficient respectively, and $\sigma(\nu,\hat{\Omega};\nu',\hat{\Omega}')$ is the scattering kernel which determines the amount of radiation intensity at a frequency ν' in a direction $\hat{\Omega}'$ being scattered into the intensity $I_{\nu}(\hat{\Omega})$. For instance, in the photon-electron scattering process, the scattering kernel is determined by the momentum distribution of the electrons and the differential scattering cross-section, the Klein-Nishna differential cross-section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm kn}=\left(\frac{e^{2}}{m_{\rm e}c^{2}}\right)^{2}\left(\frac{k_f}{k_i}\right)^{2}f(k_f,\hat{\epsilon}_f;k_i,\hat{\epsilon}_i)=\frac{3\sigma_T}{8\pi}\left(\frac{k_f}{k_i}\right)^{2}f(k_f,\hat{\epsilon}_f;k_i,\hat{\epsilon}_i)\ , \tag{2}$$

where e is the electron charge, m_e is the electron mass, σ_T is the Thomson cross section, k_i and k_f are the wave numbers of the photon before and after the scattering respectively, and $\hat{\epsilon}_i$ and $\hat{\epsilon}_i$ are the corresponding polarization vectors of the photon. The function

$$f(k_f,\hat{\epsilon}_f;k_i,\hat{\epsilon}_i)=\left|\hat{\epsilon}_f^{*}\cdot\hat{\epsilon}_i\right|^{2}+\frac{(k_f-k_1)^{2}}{4\;k_fk_i}\left[1+\left(\hat{\epsilon}_f^{*}\times\hat{\epsilon}_f\right)\cdot\left(\hat{\epsilon}_i\times\hat{\epsilon}_i^{*}\right)\right] \tag{3}$$

(see Jackson 1975).

In the absence of scattering, the covariant form of the radiative transfer equation may be written as

$$\frac{d\mathcal{I}}{d\xi}=k_{\alpha}\frac{\partial\mathcal{I}}{\partial x^{\alpha}}-\Gamma^{\alpha}_{\beta\gamma}\frac{\partial\mathcal{I}}{\partial x^{\alpha}}=k^{\alpha}u_{\alpha}\Big|_{\xi}\left[\,\eta_0-\chi_0\mathcal{I}\,\right] \tag{4}$$

(Baschek et al. 1997; Fuerst & Wu 2004; Wu et al. 2008), where $\mathcal{I}$ is the invariant intensity of the radiation, η_0 and χ_0 are the invariant emission and absorption coefficients respectively (evaluated at a local inertial frame), ξ is the Affline parameter, k^{α} is the propagation (wave number) 4-vector of the radiation, and u^{α} is the 4-velocity of the medium interacting with the radiation. Equation (4) is similar in form to equation (1) without the scattering term. The term $k^{\alpha}u_{\alpha}|_{\xi}$ is a correction factor for the aberration and energy shift in the transformation between references frames. For covariant transfer of radiation in the presence of scattering, we expect a radiative transfer equation of the form

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$$\frac{d\mathcal{I}(x^\beta,k^\beta)}{d\xi} = k^\alpha u_\alpha \Big|_\xi \left[\eta_0(x^\beta,k^\beta) - \chi_0(x^\beta,k^\beta)\mathcal{I}(x^\beta,k^\beta) + \int d^4k^\beta \; \sigma(x^\beta;k^\beta,k'^\beta)\mathcal{I}(x^\beta,k'^\beta)\right] \tag{5}$$

analogous to equation (1). Several methods have been proposed to solve the above equation or to obtain an approximate solution. For instance, one could transform the integro-differential radiative transfer equation into a set of differential equations using a moment expansion (Thorne 1981; Fuerst 2006; Wu et al. 2008). Nevertheless, one needs to specify the properties of the medium spanning the space-time. In addition to the global flow dynamics, one also needs to know how the radiation interacts with the medium, via the emission coefficient, absorption coefficient and the scattering kernel, at least in the local inertial frame. The invariant emission and absorption coefficients can be easily derived from the conventional emission and absorption coefficients (see Fuerst & Wu 2004, 2007). The derivation of the scattering kernel is more complicated. Some attempts have been made but only numerical results were obtained due to the complexity of the underlying mathematics. To date a closed-form expression for the corresponding scattering kernel is not available. The lacking of a closed-form scattering kernel hinders the development of fast and accurate numerical algorithms to solve the covariant radiative transfer equation, which itself can be numerically intensive.

In this article, we show the derivation of the invariant scattering kernel for photon-electron (Compton) scattering in a general relativistic setting and show how to obtain a closed-form expression in terms of hypergeometric functions. We organise the article as follows. In §2, 3 and 4 we present the basic physics of covariant Compton scattering, formulate the electron distribution and derive the general expression of the scattering kernel in the general relativistic radiative transfer equation. In §5 we express the scattering cross-section in moments, with which we can transform the radiative transfer equation from an integro-differential equation to a set of coupled moment differential equations. In §6, 7 and 8 we show the techniques and the relevant mathematics to obtain various components for a close form expression for the Compton scattering kernel. In §9 we show how to execute the final integration required in the whole derivation.

2 COVARIANT COMPTON SCATTERING

Here and hereafter in this article we work in geometrical units (with $G = c = h = 1$) and adopt a $(-,+,+,+)$ metric signature. Energy-momentum conservation implies that

$$k^\alpha + p^\alpha = k'^\alpha + p'^\alpha \ , \tag{6}$$

in a photon-electron scattering process. Here unprimed and primed variables denote, respectively, variables evaluated before and after scattering. The 4-momentum of a photon k^α and the 4-momentum of an electron p^α satisfy $k^\alpha k_\alpha = k'^\alpha k'_\alpha = 0$ and $p^\alpha p_\alpha = p'^\alpha p'_\alpha = -m_e^2$ respectively, where m_e is the electron mass. Energy-momentum conservation also leads to the invariance relation

$$k^\alpha p_\alpha = k'^\alpha p'_\alpha \ , \tag{7}$$

and a covariant generalised energy-shift formula for the scattered photon

$$k'^\alpha (k_\alpha + p_\alpha) = k^\alpha p_\alpha \ . \tag{8}$$

As the scattering process occurs in a relativistic fluid, the derivation of the scattering opacity due to ensembles of photons and electrons requires expressing the scattering variables of the particles in the local reference rest frame comoving with the fluid element, as well as specifying the transformation between the fluid element's rest frame and the observer's frame. We denote the fluid 4-velocity, in the fluid rest frame, as u^α and the electron 4-velocity as v^α, Clearly, $u^\alpha u_\alpha = -1$, and $v^\alpha v_\alpha \equiv v < 1$. We specify the directional unit 4-vector of the photon in the fluid rest frame as n^α, which is given by

$$n^\alpha = \frac{P^{\alpha\beta}k_\beta}{||P^{\alpha\beta}k_\beta||} \ , \tag{9}$$

where the tensor $P^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$ projects onto the 3-surface orthogonal to k_β. We may construct a variable

$$\gamma = -\frac{k^\alpha u_\alpha}{m_e} \ , \tag{10}$$

where m_e is the mass of the electron, and express n^α as

$$n^\alpha = \frac{k^\alpha}{\gamma m_e} - u^\alpha \ . \tag{11}$$

Hence, we have

$$k^\alpha = m_e\gamma(n^\alpha + u^\alpha) \ . \tag{12}$$

Similarly we construct a variable

$$\lambda = -\frac{p^\alpha u_\alpha}{m_e} \tag{13}$$

for the electrons. Obviously, $\lambda = 1/\sqrt{1-v^2}$, and it is simply the Lorentz factor of the electron. The directional 4-velocity of the electron in the fluid frame is therefore

$$\hat{v}^\alpha = \frac{P^{\alpha\beta}p_\beta}{||P^{\alpha\beta}p_\beta||} = \frac{p^\alpha - m_e\lambda u^\alpha}{m_e\lambda v} \ . \tag{14}$$

It follows that

$$v^\alpha = \frac{p^\alpha}{m_e\lambda} - u^\alpha \ . \tag{15}$$

and

$$p^\alpha = m_e\lambda(v^\alpha + u^\alpha) \ . \tag{16}$$

Note that the photon 4-momentum after the scattering is

$$k'^\alpha = m_e\gamma'(n'^\alpha + u^\alpha) \ . \tag{17}$$

Thus, we have

$$k^\alpha k'_\alpha = m_e^2\gamma\gamma'(\zeta - 1) \ , \tag{18}$$
$$p^\alpha k_\alpha = m_e^2\lambda\gamma(v^\alpha n_\alpha - 1) \ , \tag{19}$$
$$p^\alpha k'_\alpha = m_e^2\lambda\gamma'(v^\alpha n'_\alpha - 1) \ , \tag{20}$$

where $\zeta = n^\alpha n'_\alpha$ is the direction cosine of the photon deflection caused by the scattering. Hence we may express the energy-momentum conservation, equation (8), as

$$m_e^2\gamma\gamma'\left[\zeta - 1 + \lambda\left(\frac{1 - v^\alpha n_\alpha}{\gamma'} - \frac{1 - v^\alpha n'_\alpha}{\gamma}\right)\right] = 0 \ . \tag{21}$$

The cross-section of scattering of a photon by an electron is given in Kershaw et al. (1986) as

$$\sigma(\gamma \to \gamma', \hat{\Omega} \to \hat{\Omega}', \overline{v}) = \frac{3\sigma_T}{16\pi\gamma\nu\lambda}\left\{1 + \left[1 - \frac{1-\zeta}{\lambda^2 DD'}\right]^2 + \frac{(1-\zeta)^2\gamma\gamma'}{\lambda^2 DD'}\right\}\delta\left(\zeta - 1 + \lambda\left[\frac{D}{\gamma'} - \frac{D'}{\gamma}\right]\right) \ , \tag{22}$$

where $D \equiv 1 - \hat{\Omega}\cdot\overline{v}/c = 1 - v^\alpha n_\alpha$. Using equations (18), (19) and (20), we may express the photon-electron scattering cross-section, equation (22), in a covariant form:

$$\sigma(\gamma \to \gamma', n^\alpha \to n'^\alpha, \overline{v}) = \frac{3\sigma_T}{16\pi\gamma\nu\lambda}\left\{1 + \left(1 + \frac{m_e^2 t}{k^\alpha k'_\alpha}\right)^2 + t\right\}\delta\left(\frac{\mathcal{P}}{m_e^2\gamma\gamma'}\right) \ , \tag{23}$$

with

$$t = \frac{(k^\alpha k'_\alpha)^2}{(p^\alpha k_\alpha)(p^\beta p'_\beta)} \ , \tag{24}$$

and

$$\mathcal{P} = k^\alpha k'_\alpha + p^\alpha k'_\alpha - p^\alpha k_\alpha \ . \tag{25}$$

This cross-section, equation (23), will be integrated over a relativistic electron distribution function to yield the scattering kernel.

3 ELECTRON DISTRIBUTION FUNCTION

The energy of an electron is $E = \lambda m_e c^2$, and the linear momentum is $p = \lambda m_e v$, which gives

$$\frac{dp}{dv} = m_e\frac{d}{dv}(\lambda v) = m_e\lambda^3 \ . \tag{26}$$

For an ensemble of relativistic electrons with isotropic momenta, the distribution function is

$$\Psi(\overline{p}) = Ce^{-E(\overline{p})/k_B T_e} \ , \tag{27}$$

where E is the electron energy, T_e the electron temperature, k_B the Boltzmann constant and C is the normalisation constant. Note that the distributions of electrons in momentum space and in velocity space are related by

$$f(\overline{v})v^2 dv = \Psi(\overline{p})p^2 dp \ . \tag{28}$$

With the Jacobian transformation, we may express the above as

$$f(\overline{v}) = \frac{p^2}{v^2}\frac{dp}{dv}\Psi(\overline{p}) \ . \tag{29}$$

It immediately follows that

$$f(\overline{v}) = C'\lambda(v)^5 e^{-\lambda(v)/\tau} \ , \tag{30}$$

where $C' = m_e^3 C$ is a constant and $\tau = k_B T_e/m_e$. The normalisation of the distribution function $f(\overline{v})$ to unity, i.e.

$$\int d\overline{v} f(\overline{v}) = 4\pi \int_0^1 dv\, v^2 f(\overline{v}) = 1 \ , \tag{31}$$

gives a relativistic Maxwellian distribution

$$f(\overline{v}) = \frac{\lambda(v)^5 e^{-\lambda(v)/\tau}}{4\pi\tau K_2(1/\tau)} \ , \tag{32}$$

where $K_2(...)$ is the modified Bessel function of the second kind.

4 COMPTON SCATTERING KERNEL

The Compton scattering kernel is determined by the integration of the photon-electron scattering cross-section convolved with the electron velocity distribution function over the electron velocity, i.e.

$$\sigma_s(\gamma \rightarrow \gamma', \zeta, \tau) = \frac{3\rho\sigma_T}{16\pi\gamma\nu} \int d\overline{v}\ \frac{f(v)}{\lambda} \left\{ 1 + \left(1 + \frac{m_e^2 t}{k^\alpha k'_\alpha} \right)^2 + t \right\} \delta\left(\frac{\mathcal{P}}{m_e^2 \gamma\gamma'} \right) \ . \tag{33}$$

To evaluate the above integral, we first split the argument in the delta function into two terms (see Beason et al. 1991):

$$\delta\left(\frac{\mathcal{P}}{m_e^2 \gamma\gamma'} \right) \longrightarrow \delta\left(\Gamma + w\, \hat{v}^\alpha \hat{w}_\alpha \right) \ . \tag{34}$$

Here, $\Gamma = \zeta - 1 + \lambda(\gamma'^{-1} - \gamma^{-1})$, and $\hat{v}^\alpha$ is given in equation (14). Note that

$$w_\alpha = \frac{\lambda v}{\gamma\gamma'}(\gamma' n'_\alpha - \gamma n_\alpha) \ . \tag{35}$$

It follows that

$$w \equiv ||w^\alpha w_\alpha|| = \frac{\lambda v q}{\gamma\gamma'} \ , \tag{36}$$

and

$$\hat{w}_\alpha = \frac{\gamma' n'_\alpha - \gamma n_\alpha}{q} \ , \tag{37}$$

where

$$q = \sqrt{\gamma^2 + \gamma'^2 - 2\gamma\gamma'\zeta} \ . \tag{38}$$

The delta function in equation (33) then becomes

$$\delta(\Gamma + w\, \hat{v}^\alpha \hat{w}_\alpha) = \frac{1}{w}\ \delta\left(\frac{\Gamma}{w} + \hat{v}^\alpha \hat{w}_\alpha \right) \ , \tag{39}$$

which implies that he condition that

$$\hat{v}^\alpha \hat{w}_\alpha = -\frac{\Gamma}{w} \tag{40}$$

must be satisfied for the scattering process to occur.

As the magnitude of the product of two unit vectors is always smaller than unity, $||\hat{v}^\alpha \hat{w}_\alpha|| \leqslant 1$, and hence $|| - \Gamma/w|| \leqslant 1$. This leads to an inequality

$$(1 - \zeta) + \lambda(\gamma^{-1} - \gamma'^{-1}) \leqslant \frac{\lambda v q}{\gamma\gamma'} \ , \tag{41}$$

which is akin to solving the quadratic equation $A\lambda^2 - B\lambda - C = 0$, where the coefficients A, B and C are given by

$$A = 2\gamma\gamma'(1 - \zeta) \ , \tag{42}$$
$$B = 2\gamma\gamma'(1 - \zeta)(\gamma' - \gamma) \ , \tag{43}$$
$$C = q^2 + (\gamma\gamma'(1 - \zeta))^2 \ . \tag{44}$$

We take the positive solution to this quadratic equation, that is

$$\lambda_+ = \left(\frac{\gamma' - \gamma}{2} \right) + \frac{q}{2}\sqrt{1 + \frac{2}{\gamma\gamma'(1 - \zeta)}} \ , \tag{45}$$

which is the minimum value of λ. We have used the delta function to fix $\hat{v}^\alpha \hat{w}_\alpha$, and from it have established the form of λ as a function of ζ, which is crucial in later calculations involving integrations over (λ, ζ). We may now perform the integral in equation (33) using

$$\int d\overline{v} = \int_0^1 dv\ v^2 \int_{-1}^1 d(\hat{v}^\alpha \hat{w}_\alpha) \int_0^{2\pi} d\phi\ . \tag{46}$$

We note, as in Kershaw et al. (1986), the angular addition formula

$$\hat{v}^\alpha \hat{m}_\alpha = (\hat{n}^\alpha \hat{m}_\alpha)(\hat{v}^\alpha \hat{n}_\alpha) + \sqrt{1-(\hat{n}^\alpha \hat{m}_\alpha)^2}\sqrt{1-(\hat{v}^\alpha \hat{n}_\alpha)^2}\cos\phi, \tag{47}$$

where $\hat{m}_\alpha$ is equal to $\hat{w}_\alpha$ or $\hat{w}'_\alpha$, the photon velocity before or after collision respectively. It is easily verified that

$$n^\alpha \hat{w}_\alpha = \frac{\gamma'\zeta - \gamma}{q}\ , \tag{48}$$

$$n'^\alpha \hat{w}_\alpha = \frac{\gamma' - \gamma\zeta}{q}\ , \tag{49}$$

$$\hat{v}^\alpha \hat{w}_\alpha = \frac{\gamma\gamma'(1-\zeta) + \lambda(\gamma'-\gamma)}{q\lambda v}\ . \tag{50}$$

As such in equation (46) we need only considering the ϕ integral. The curly-bracketed term may be rewritten as

$$\left\{1 + \left(1 + \frac{m_e^2 t}{k^\alpha k'_\alpha}\right)^2 + t\right\} = 2 + \left[\gamma\gamma'(1-\zeta) - 2 - \frac{2}{\gamma\gamma'(1-\zeta)}\right]\left\{(\lambda\gamma' D')^{-1} - (\lambda\gamma D)^{-1}\right\} + (\lambda\gamma' D')^{-2} + (\lambda\gamma D)^{-2}\ , \tag{51}$$

where $D = 1 - v^\alpha n_\alpha$, as in equation (22), and equation (51) must be integrated term-by-term over ϕ. The integrals to solve have the forms:

$$I_1 = \int_0^{2\pi} \frac{d\phi}{\alpha + \beta\cos\phi}\ , \tag{52}$$

$$I_2 = \int_0^{2\pi} \frac{d\phi}{(\alpha + \beta\cos\phi)^2}\ , \tag{53}$$

where

$$\alpha = 1 - v(\hat{w}^\alpha n_\alpha)(\hat{w}^\alpha \hat{v}_\alpha)\ , \tag{54}$$

$$\beta = -v\sqrt{1-(\hat{w}^\alpha n_\alpha)^2}\sqrt{1-(\hat{w}^\alpha \hat{v}_\alpha)^2}\ . \tag{55}$$

Clearly, the two integrals are related, via

$$I_2 = -\frac{dI_1}{d\alpha}\ , \tag{56}$$

and therefore we need only to evaluate I_1. It can be verified that

$$I_1 = \frac{2\pi}{(\alpha^2 - \beta^2)^{1/2}}\ , \tag{57}$$

$$I_2 = \frac{2\pi\alpha}{(\alpha^2 - \beta^2)^{3/2}}\ . \tag{58}$$

Also, we have

$$\alpha = \frac{\gamma\gamma'^2}{\lambda q^2}\left[(\lambda+\gamma)(\gamma^{-1} + \gamma'^{-1}) - (1+\zeta)\right]\ , \tag{59}$$

$$\alpha' = \frac{\gamma^2\gamma'}{\lambda q^2}\left[(\lambda-\gamma')(\gamma^{-1} + \gamma'^{-1}) - (1+\zeta)\right]\ , \tag{60}$$

$$\beta = \frac{\gamma'\omega(\zeta-1)}{\lambda q^2}\sqrt{A\lambda^2 - B\lambda - C}\ , \tag{61}$$

$$\beta' = \left(\frac{\gamma}{\gamma'}\right)\beta\ , \tag{62}$$

$$\alpha^2 - \beta^2 = \frac{\gamma'^2(1-\zeta)^2\left[(\gamma+\lambda)^2 + \omega^2\right]}{\lambda^2 q^2}\ , \tag{63}$$

$$\alpha'^2 - \beta'^2 = \frac{\gamma^2(1-\zeta)^2\left[(\gamma'-\lambda)^2 + \omega^2\right]}{\lambda^2 q^2}\ , \tag{64}$$

where $\omega^2 = (1+\zeta)/(1-\zeta)$. Therefore,

$$\int_0^{2\pi} d\phi\ D^{-1} = \frac{2\pi\lambda q}{\gamma'} \frac{(1-\zeta)^{-1}}{[(\gamma+\lambda)^2+\omega^2]^{1/2}} , \quad (65)$$

$$\int_0^{2\pi} d\phi\ D'^{-1} = \frac{2\pi\lambda q}{\gamma} \frac{(1-\zeta)^{-1}}{[(\gamma'-\lambda)^2+\omega^2]^{1/2}} , \quad (66)$$

$$\int_0^{2\pi} d\phi\ D^{-2} = \frac{2\pi\gamma\lambda^2 q}{\gamma'(1-\zeta)^2} \frac{\left[(\gamma+\lambda)(\gamma^{-1}+\gamma'^{-1})-(1+\zeta)\right]}{[(\gamma+\lambda)^2+\omega^2]^{3/2}} , \quad (67)$$

$$\int_0^{2\pi} d\phi\ D'^{-2} = \frac{2\pi\gamma'\lambda^2 q}{\gamma(1-\zeta)^2} \frac{\left[(\gamma'-\lambda)(\gamma^{-1}+\gamma'^{-1})-(1+\zeta)\right]}{[(\gamma'-\lambda)^2+\omega^2]^{3/2}} . \quad (68)$$

We may now rewrite the cross-section in equation (33) as

$$\sigma_s(\gamma\to\gamma',\zeta,\tau) = \frac{3\rho\sigma_T}{8\gamma\nu}\int_{\lambda_+}^{\infty} d\lambda\ \frac{f(\lambda)}{\lambda^5}\left\{\frac{2\gamma\gamma'}{q} + R(\gamma+\lambda) - R(\gamma'-\lambda)\right\} , \quad (69)$$

where the function $R(...)$ is given by

$$R(x) \equiv \frac{\left(x\left(\gamma^{-1}+\gamma'^{-1}\right)-1\right)-\zeta}{(1-\zeta)^2(x^2+\omega^2)^{3/2}} + \left[-\gamma\gamma' + \frac{2}{1-\zeta} + \frac{2}{\gamma\gamma'(1-\zeta)^2}\right]\frac{1}{(x^2+\omega^2)^{1/2}} . \quad (70)$$

This expression is more symmetric than the equivalent expression for the cross-section obtained by Prasad et al. (1986).

5 ANGULAR MOMENTS OF THE COMPTON CROSS SECTION

In solving the full radiative transfer equation with Compton scattering we can use a generalised Eddington Approximation to compute successive angular moment integrals of σ_s. We want to solve integrals of the form

$$\int d\zeta\ \zeta^n \sigma_s(\gamma\to\gamma',\zeta,\tau) = \frac{3\rho\sigma_T}{8\gamma\nu}\int_{-1}^{1} d\zeta \int_{\lambda_+}^{\infty} d\lambda\ \frac{f(\lambda)}{\lambda^5}\left\{\frac{2\gamma\gamma'}{q} + R(\gamma+\lambda) - R(\gamma'-\lambda)\right\} . \quad (71)$$

However, as equation (71) stands, integrating over $f(\lambda)$ and λ proves impossible. Somewhat ironically, the solution to this problem is to apply the method of Brinkmann (1984), and change the order of integration, after the aforementioned method of Prasad et al. At the time, neither group saw that combining both of their methods would allow them to overcome the inherent algebraic and mathematical difficulties of the problem.

Essentially we have a function $\lambda_+(\zeta)$ which we seek to invert, i.e. find $\zeta(\lambda_+)$.

$$\lambda_+(-1) \equiv \lambda_L = \frac{\gamma'-\gamma}{2} + \frac{\gamma'+\gamma}{2}\sqrt{1+\frac{1}{\gamma\gamma'}} , \quad (72)$$

whereas

$$\lim_{\zeta\to 1} \lambda_+ = +\infty . \quad (73)$$

We now seek the minimum value of λ_+, that is we seek the value of ζ such that λ_+ is minimised. It is readily shown that

$$\zeta_{1,2} = 1 \pm \left(\gamma^{-1}-\gamma'^{-1}\right) , \quad (74)$$

and hence

$$\lambda_{min} = 1 + \frac{1}{2}\Big[(\gamma'-\gamma) + |\gamma'-\gamma|\Big] . \quad (75)$$

Normally $\lambda_{min} < \lambda_L$ by definition. However, $\lambda_L \geqslant \lambda_{min}$ if the following condition is satisfied

$$|\gamma^{-1}-\gamma'^{-1}| \geqslant 2, \quad (76)$$

which essentially states that if the photon gains sufficient energy post collision then the aforementioned inequality is satisfied. Finally, we must rearrange λ_+ to find ζ as a function of λ (and γ, γ'). After much algebra one finds

$$\zeta_\pm = 1 - \frac{1}{\gamma\gamma'}\left[\lambda(\gamma-\gamma') + (\lambda^2-1) \mp \sqrt{\lambda^2-1}\sqrt{(\lambda+\gamma-\gamma')^2-1}\right] . \quad (77)$$

We now may state the following identity

$$\int_{-1}^{1} d\zeta \int_{\lambda_+}^{\infty} d\lambda = \int_{\lambda_L}^{\infty} d\lambda \int_{-1}^{\zeta_+} d\zeta + \int_{\lambda_{min}}^{\lambda_L} d\lambda \int_{\zeta_-}^{\zeta_+} d\zeta , \quad (78)$$

which, if $\lambda_L \leqslant \lambda_{min}$ then $\lambda_L = \lambda_{min}$ and so the second term in equation (78) vanishes and we need only evaluate the first double integral.

6 PERFORMING THE INTEGRATIONS OVER ζ

In evaluating equation (71) with equation (78), we must solve three different types of integral

$$Q_n = \int d\zeta \, \frac{\zeta^n}{q} \,, \tag{79}$$

$$R_n = \int \frac{d\zeta \, \zeta^n}{(1-\zeta)^2 \left(x^2 + \frac{1+\zeta}{1-\zeta}\right)^{3/2}} \,, \tag{80}$$

$$S_{n,m} = \int \frac{d\zeta \, \zeta^n}{(1-\zeta)^m \left(x^2 + \frac{1+\zeta}{1-\zeta}\right)^{1/2}} \,, \qquad (m = 0, 1, 2) \,. \tag{81}$$

Note an identity useful for later working

$$\frac{dS_{n,2}}{dx} = -xR_n \,. \tag{82}$$

Thus, we may write in more compact notation

$$\mathcal{M}_n = \int d\zeta \, \zeta^n \left\{ \frac{2\gamma\gamma'}{q} + R(\gamma+\lambda) - R(\gamma'-\lambda) \right\} \tag{83}$$

$$= A_n + B_n(\gamma+\lambda) - B_n(\gamma'-\lambda) \,, \tag{84}$$

where

$$A_n = 2\gamma\gamma' Q_n, \tag{85}$$

$$B_n = (\delta R_n - R_{n+1}) + \left(-\gamma\gamma' S_{n,0} + 2S_{n,1} + \frac{2}{\gamma\gamma'} S_{n,2}\right) \,, \tag{86}$$

where

$$\delta = x(\gamma^{-1} + \gamma'^{-1}) - 1. \tag{87}$$

In equations (80) and (81), the integrals have an x-dependence which is crucial to their evaluation. As noted earlier, $x \equiv (\gamma+\lambda)$ or $x \equiv (\gamma'-\lambda)$, depending on whether we are dealing with pre- or post-collision integrals. Regardless, the evaluation of these integrals yields different results depending on whether $x^2 < 1$, $x^2 = 1$ or $x^2 > 1$.

We now state the $n = 0, 1, 2$ moments for A_0 and B_0.

<u>For $x^2 \neq 1$:</u>

$$A_0 = -2q \,, \tag{88}$$

$$A_1 = -\frac{2q}{3\gamma\gamma'}(q^2 + 3\gamma\gamma'\zeta) \,, \tag{89}$$

$$A_2 = -\frac{2q}{15\gamma^2\gamma'^2}\left[2(\gamma^2+\gamma'^2)(q^2+3\gamma\gamma'\zeta) + 3\gamma^2\gamma'^2\zeta^2\right] \,, \tag{90}$$

$$B_0 = b\left[2 - x(\gamma^{-1}+\gamma'^{-1}) + \frac{2(1+\gamma\gamma')}{x^2-1} + \gamma\gamma'(1-\zeta) + \frac{2}{\gamma\gamma'}(x^2+\omega^2)\right] + \frac{2(2x^2+\gamma\gamma'-1)}{1-x^2}C(x) \,, \tag{91}$$

$$B_1 = \frac{b}{1-x^2}\left[x(1+x^2)(\gamma^{-1}+\gamma'^{-1}) - (7+\zeta) + 2x^2(\zeta-2) + \left\{(1+x^2)+(1-x^2)\zeta\right\}\left\{\frac{4(1-x^2)-\gamma^2\gamma'^2(1-\zeta^2)}{2\gamma\gamma'(1-\zeta)}\right\}\right] + \left[\frac{2(\delta-2x^2)}{x^2-1} + \frac{(2-\gamma\gamma')(2x^2+1)}{(x^2-1)^2} + \frac{4}{\gamma\gamma'}\right]C(x) \,, \tag{92}$$

$$B_2 = \frac{b}{(1-x^2)^2}\Bigg[-\delta(2+\zeta+x^2(3-\zeta)+x^4) + (1+\zeta+x^2(1-\zeta))(x^2(3+\zeta)-\zeta) + \frac{6}{x^2-1}\Big\{(4+9x+2x^4)(3+3x^2+\gamma\gamma') + (x^2-1)(3(\gamma\gamma'-1)+2x^2(\gamma\gamma'-6))\zeta + (x^2-1)^2(2\gamma\gamma'-3)\zeta^2\Big\} + \frac{2(1-x^2)((1+\zeta)+x^2(1-\zeta))(2-x^2-\zeta)}{\gamma\gamma'(1-\zeta)}\Bigg] + \frac{1}{1-x^2}\left[\frac{2(\delta(2x^2+1)-(2x^4+1))}{1-x^2} + \frac{(3+\gamma\gamma'(5+6(x^2-1)+2(x^2-1)^2))}{(1-x^2)^2} + \frac{4(1-2x^2)}{\gamma\gamma'}\right]C(x) \,, \tag{93}$$

where

$$b = \frac{\sqrt{1-\zeta}}{\sqrt{(1+x^2)+(1-x^2)\zeta}} \ . \tag{94}$$

Also, the function $C(x)$ is defined as

$$C(x) = \frac{1}{\sqrt{1-x^2}} \text{ Arctan} \left(\frac{\sqrt{1-x^2}\sqrt{1-\zeta}}{\sqrt{(1+x^2)+\zeta(1-x^2)}} \right), \text{ if } x^2 < 1 \ , \tag{95}$$

$$C(x) = \frac{1}{\sqrt{1-x^2}} \text{ Arcsinh} \left(\frac{\sqrt{x^2-1}\sqrt{1-\zeta}}{\sqrt{2}} \right), \text{ if } x^2 > 1 \ . \tag{96}$$

Note that equations (88) $-$ (90) are independent of x.
For $x^2 = 1$:

$$B_0 = \sqrt{\frac{1-\zeta}{2}} \left[\frac{4}{\gamma\gamma'(1-\zeta)} - 4 - \delta + \frac{2\gamma\gamma'(1-\zeta)}{3} + \frac{2+\zeta}{3} \right] \ , \tag{97}$$

$$B_1 = \sqrt{\frac{1-\zeta}{2}} \left[\frac{4(2-\zeta)}{\gamma\gamma'}(1-\zeta) - \frac{4}{3}(2+\zeta) - \frac{\delta(2+\zeta)}{3} + \frac{2\gamma\gamma'(1-\zeta)(2+3\zeta)}{15} + \frac{8+4\zeta+3\zeta^2}{15} \right] \ , \tag{98}$$

$$B_2 = \sqrt{\frac{1-\zeta}{2}} \left[\frac{4(8-4\zeta-\zeta^2)}{3\gamma\gamma'(1-\zeta)} - \frac{(4+\delta)(8+4\zeta+3\zeta^2)}{15} + \frac{2\gamma\gamma'(1-\zeta)(8+12\zeta+15\zeta^2)}{105} + \frac{16+8\zeta+6\zeta^2+5\zeta^3}{35} \right] \ . \tag{99}$$

We have now found $\mathcal{M}_n$ for $n = 0$, 1 and 2, for all values of x. In principle one can integrate equation (79), (80) and (81) to arbitrary n, but, as seen in equations (88) - (99) the resultant algebraic expressions become extremely cumbersome. Moreover, the expressions for A_n must be evaluated once per scattering event, and B_n twice per scattering event. Given the inherent algebraic complexity this will likely lead to significant loss of precision, in particular between cancellations of terms of similar value or of particular smallness.

Therefore, we may write the Compton scattering kernel as

$$\sigma_{sn}(\gamma \to \gamma', \tau) = \frac{3\rho\sigma_T}{8\gamma\nu} \left[\int_{\lambda_L}^{\infty} d\lambda \ \frac{f(\lambda)}{\lambda^5} \mathcal{M}_n \Big|_{-1}^{\zeta_+} + \int_{\lambda_{min}}^{\lambda_L} d\lambda \ \frac{f(\lambda)}{\lambda^5} \mathcal{M}_n \Big|_{\zeta_-}^{\zeta_+} \right] \ , \tag{100}$$

where, as noted before, the second term in square brackets in the above equation vanishes when $\lambda_L \leqslant \lambda_{min}$, saving significant computational expense. In the case of $x^2 = 1$, the integrals inside the bracket can be transformed from $\lambda \to \gamma$ or γ'. Given the algebraic complexity of equations (88) - (99) this is as far as we can proceed analytically. The integration over λ will have to be performed with an appropriate numerical scheme.

Naturally, the question arises as to whether the integrals in equation (100) can be performed analytically. As it stands, the method presented thus far would likely require arbitrary precision arithmetic to evaluate, and therefore be computationally expensive and time consuming - and all for just one scattering event! In the next section we discuss how to evaluate the integrals, equation (79)-(81), more rapidly, and for arbitrary n.

7 EVALUATING THE MOMENT INTEGRALS FOR ARBITRARY N

In the previous section we derived analytic expressions for the first three moments of the Compton scattering kernel. As the order of the moments increased, the algebraic complexity of the resultant expression increased rapidly. Clearly the method, as it stands, does not lend itself readily to the evaluation of higher-order moments, necessary for more accurate evaluation of radiation transport problems. A much faster method is to evaluate equations (76)-(78) recursively, which is detailed in this section.

First, let us consider equation (75)

$$Q_{n+1} = \int d\zeta \frac{\zeta^{n+1}}{q}. \tag{101}$$

By employing the identity $\frac{dq}{d\zeta} = -\frac{\gamma\gamma'}{\zeta}$, upon integrating equation (101) by parts, we find the following recurrence relation

$$\gamma\gamma'(2n+3)Q_{n+1} = (\gamma^2 + \gamma'^2)(n+1)Q_n - q\zeta^{n+1} \ . \tag{102}$$

Thus with the seed $Q_0 = -q/\gamma\gamma'$, Q_n may be evaluated for arbitrary n. We next consider equation (79) in the form

$$R_n = -\frac{1}{2\sqrt{2}} \int du \, u^{-1/2}(1-u)^n(1-cu)^{-3/2} \ , \tag{103}$$

where we have used the substitution $u = 1-\zeta$ and $c \equiv (1-x^2)/2$. By expanding in series the term $(1-u)^n$ we may write equation (103) as

$$R_n = \sum_{k=0}^{n} \frac{(-1)^{k+1}\binom{n}{k}}{2\sqrt{2}} \int du \frac{u^{k-1/2}}{(1-cu)^{3/2}} \ . \tag{104}$$

Defining the integral

$$I_R(k+1) = \int du \frac{u^{k+1/2}}{(1-cu)^{3/2}}, \tag{105}$$

a recursion relation for equation (105) may be found by integrating by parts

$$2kc\, I_R(k+1) = (2k+1) I_R(k) - \frac{2u^{k+1/2}}{\sqrt{1-cu}} \ . \tag{106}$$

The value $I_R(0)$ immediately follows, but to perform recursively we need the seed value

$$I_R(1) = \frac{2\sqrt{u}}{c\sqrt{1-cu}} - \frac{2}{c^{3/2}} \arcsin(\sqrt{cu}) \ . \tag{107}$$

Thus we may define R_n as

$$R_n = \sum_{k=0}^{n} \frac{(-1)^{k+1}\binom{n}{k}}{2\sqrt{2}} I_R(k) \ , \tag{108}$$

which can be solved for arbitrary n with the initial seed values. We apply a similar process for $S_{n,m}$:

$$S_{n,m} = \sum_{k=0}^{n} \frac{(-1)^{k+1}\binom{n}{k}}{\sqrt{2}} I_S(k,m) \ , \tag{109}$$

where

$$I_S(k+1,m) = \int du \frac{u^{k+1-m}}{\sqrt{1-cu}} \ . \tag{110}$$

After some working we obtain the recursion relation

$$(2k-2m+3)c\, I_S(k+1,m) = 2(k+1-m) I_S(k,m) - 2u^{k+1-m}\sqrt{1-cu} \ . \tag{111}$$

This identity requires three different seed values for the cases $m = 0$, 1 and 2 respectively:

$$I_S(0,0) = -\frac{2\sqrt{1-cu}}{c} \ , \tag{112}$$

$$I_S(0,1) = -2\,\mathrm{arctanh}(\sqrt{1-cu}) \ , \tag{113}$$

$$I_S(0,2) = -\frac{\sqrt{1-cu} + cu\,\mathrm{arctanh}(\sqrt{1-cu})}{u} \equiv \frac{c}{2u} I_S(0,0) + \frac{c}{2} I_S(0,1) \ . \tag{114}$$

Thus equations (102), (108) and (109) allow us to solve (79)-(81) iteratively. In computing angular moments of the Klein-Nishina cross-section this will greatly reduce the computational time and resources required. Instead of relying on a lengthy and cumbersome algebraic expression for each moment integral, we can very rapidly calculate each moment integral using the stored numerical value of the previous moment recursively. Unfortunately, as the order increases, there will inevitably be loss of precision through subtractions in the recursion relations. Further, it is impossible to perform the final integral over the electron distribution function without either an algebraic expression for each moment, which is analytically intractable in the first instance, or an appropriate closed-form expression for each moment in terms of more generalised functions. In the following sections we detail such a method.

8 EVALUATING MOMENT INTEGRALS - HYPERGEOMETRIC FUNCTION METHOD

In this section we show how to evaluate the moment integrals in equations (79)-(81) in terms of more general functions. We briefly introduce the concept of hypergeometric functions of one and two variables (Bateman 1955) and show how the problem of relativistic Compton scattering may be greatly simplified in terms of these more general functions. Hypergeometric functions are a very general class of functions which contain many of the known mathematical functions as special or limiting cases (Luke 1969; Abramowitz & Stegun 1972) and we will use their general properties to simplify the expression and facilitate the solution of integrals of the Klein-Nishina cross-section.

8.1 Appell Hypergeometric Function Method

The Appell F_1 hypergeometric function is one of a set of four hypergeometric series of two variables (Appell 1880; Appell & Kampé de Fériet 1926). It is a very general class of special function, containing many other special functions as particular or limiting cases, including hypergeometric functions of one variable like the Gauss ${}_2\mathrm{F}_1$ we will encounter in the next subsection. The Appell F_1 function is defined by the series expansion

$$\mathrm{F}_1[\alpha;\beta_1,\beta_2,\gamma;z_1,z_2] = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(\alpha)_{k+l}(\beta_1)_k(\beta_2)_l}{(\gamma)_{k+l}\,k!\,l!} z_1^k\, z_2^l \ , \tag{115}$$

where the notation

$$(\alpha)_n \equiv \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \ , \tag{116}$$

is the rising factorial or Pochhammer symbol. This series is absolutely convergent for $|z_1| < 1$, $|z_2| < 1$. Cases outside of the unit disc of convergence can be calculated through analytic extension of the Appell F_1 hypergeometric function (Olsson 1964), hence an algorithm can be constructed to evaluate the function numerically (e.g. Colavecchia et al. 2001; Colavecchia & Gasaneo 2004).

Consider R_n and $S_{n,m}$ (Q_n can be evaluated in terms of lower order Gauss hypergeometric functions - see next subsection). We state the solution to this integral in terms of the Appell F_1 function

$$\begin{aligned} \mathrm{R}_n &= a^{-3/2}\int d\zeta\zeta^n(1-\zeta)^{-1/2}\left(1+\frac{b}{a}\zeta\right)^{-3/2} \\ &= a^{-3/2}\int d\zeta\zeta^n\left\{\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k}{k!}\zeta^k\right\}\left\{\sum_{l=0}^{\infty}\frac{(-1)^l(\frac{3}{2})_l}{l!}\left(\frac{b}{a}\right)^l\zeta^l\right\} \ , \end{aligned} \tag{117}$$

where $a \equiv 1+x^2$, $b \equiv 1-x^2$. Performing the integral over ζ and making use of the identity

$$\frac{n+1}{n+k+l+1} = \frac{(n+1)_{k+l}}{(n+2)_{k+l}} \tag{118}$$

we obtain a closed-form expression for the moment integral R_n

$$R_n = \frac{\zeta^{n+1}}{(n+1)(1+x^2)^{3/2}}\mathrm{F}_1\left[n+1;\frac{1}{2},\frac{3}{2};n+2;\zeta,\left(\frac{x^2-1}{x^2+1}\right)\zeta\right] \ . \tag{119}$$

By the same process we obtain the closed-form expression for the moment integral $S_{n,m}$ as

$$\mathrm{S}_{n,m} = \frac{\zeta^{n+1}}{(n+1)(1+x^2)^{1/2}}\mathrm{F}_1\left[n+1;m-\frac{1}{2},\frac{1}{2};n+2;\zeta,\left(\frac{x^2-1}{x^2+1}\right)\zeta\right] \ . \tag{120}$$

In the case $x^2 = 1$ these expressions simplify to Gauss hypergeometric functions of one variable (see section 8.3). As expected from the integral expressions for the moment integrals in equations (80) and (81), (119) and (120) are identical in argument and differ only in their parameters (β_1, β_2) and a multiplicative factor $(1+x^2)$. For both of these expressions the arguments lie on or within the unit disc, hence we need only concern ourselves with two exceptional cases, namely where $\zeta = \pm 1$. The case $\zeta = 1$ (undeflected photon) reduces to a product of Gauss hypergeometric functions (Abramowitz & Stegun 1972) and the case $\zeta = -1$ (180° deflection) is solved through a simple analytic continuation (Olsson 1964). The numerical evaluation of these functions is discussed in more detail in section 9.

8.2 Gauss Hypergeometric Function Method - $x^2 \neq 1$ case

The hypergeometric function (or Gaussian hypergeometric function) is a special function which includes many other special functions as specific or limiting cases. It is defined, for $|z| < 1$, by the series

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n\,n!}z^n \ . \tag{121}$$

With this definition in mind the integrals Q_n, R_n and $S_{n,m}$ may be solved. In fact, we have done part of the work already by writing R_n and $S_{n,m}$ in summation form in equations (108) and (109). We will state the results of these integrals in terms of Gauss hypergeometric functions. In a similar fashion to the derivation of equations (119) and (120) we find, upon employing the series expansion $(1-u)^n = \sum_{k=0}^{\infty}(-1)^k\binom{n}{k}u^k$, the following

$$Q_n = \frac{\zeta^{n+1}}{(n+1)\sqrt{\gamma^2+\gamma'^2}}\,{}_2\mathrm{F}_1\left[\frac{1}{2},n+1;n+2;\frac{2\gamma\gamma'}{\gamma^2+\gamma'^2}\zeta\right] \ , \tag{122}$$

$$R_n = \sum_{k=0}^{n}\frac{(-1)^{k+1}\binom{n}{k}u^{k+\frac{1}{2}}}{\sqrt{2}(2k+1)}\,{}_2\mathrm{F}_1\left[\frac{3}{2},k+\frac{1}{2};k+\frac{3}{2};\frac{1}{2}(1-x^2)(1-\zeta)\right] \ , \tag{123}$$

$$S_{n,m} = \sum_{k=0}^{n}\frac{(-1)^{k+1}\binom{n}{k}u^{k-m+\frac{3}{2}}}{\sqrt{2}(k-m+\frac{3}{2})}\,{}_2\mathrm{F}_1\left[\frac{1}{2},k-m+\frac{3}{2};k-m+\frac{5}{2};\frac{1}{2}(1-x^2)(1-\zeta)\right] \ . \tag{124}$$

A few notes about the continuity of expressions (122)-(124). Q_n is always within the convergence region, and only lies on the boundary in the case of a perfectly elastic collision (Thomson scattering). Equations (116) and (117) can be divided into two cases: those which lie within the convergence region ($|z| < 1$) and those that lie on the boundary or outside it ($z \leqslant -1$). The case $z \leqslant -1$ (i.e. $\zeta \geqslant \frac{x^2+1}{x^2-1}$) may be solved by analytic extension with the following expression

$$ {}_2\mathrm{F}_1[a, b; b+1, z] = (1-z)^{-a} {}_2\mathrm{F}_1\left[a, 1; b+1; \frac{z}{z-1}\right] , \tag{125} $$

which brings (122)-(124) into the region $(\frac{1}{2}, 1]$ which lies with the unit disc and hence the convergence region. The Gauss hypergeometric function is well documented in the literature and there exist several codes in FORTRAN which can evaluate it both accurately and rapidly (e.g. Forrey 1997; Zhang & Jin 1996), and can handle all cases of differences of parameters and values which can give rise to numerical problems (Zhang & Jin 1996).

8.3 Gauss Hypergeometric Function Method - $x^2 = 1$ case

In the special case $x^2 = 1$ we simply evaluate equations (119) and (120) in this case. By employing the following identity

$$ \mathrm{F}_1[\alpha; \beta_1, \beta_2; \gamma; x, 0] = {}_2\mathrm{F}_1[\alpha, \beta_1; \gamma; x] , \tag{126} $$

we may express R_n and $S_{n,m}$ in the case $x = 1$ as

$$ R_n = \frac{\zeta^{n+1}}{2\sqrt{2}(n+1)} {}_2\mathrm{F}_1\left[\frac{1}{2}, n+1; n+2; \zeta\right] , \tag{127} $$

$$ S_{n,m} = \frac{\zeta^{n+1}}{\sqrt{2}(n+1)} {}_2\mathrm{F}_1\left[m - \frac{1}{2}, n+1; n+2; \zeta\right] . \tag{128} $$

Thus we have now defined the moment integrals for all values of x in closed. The final step in computing the relativistic Klein-Nishina cross-section is integrating over the electron distribution function.

9 INTEGRATING OVER THE ELECTRON DISTRIBUTION FUNCTION

To our knowledge, in all of the literature at present only integration over four of the five integration variables $(v, \hat{v}^\alpha \hat{w}_\alpha, \phi, \zeta, \lambda)$ has been performed analytically - generally a choice must be made between performing integrals of the angular moments or integrating over the electron distribution function. Note we do not consider the photon energy ν, the sixth and final integral, as this is performed numerically during the radiative transfer calculations at each point along a ray. Regardless, with the methods at present, one is left with at best two further sets of integrals to evaluate. Further, the problem as formulated in the current literature (Prasad et al. 1986; Nagirner & Poutanen 1993; Poutanen & Vurm 2010) is algebraically cumbersome. Further, it is common to resort to Monte-Carlo methods to solve the multi-dimensional integrals, as well as numerous underlying assumptions (need references here). To have a closed-form solution to the first five integrals, including the electron distribution function, would eliminate the need for evaluating multi-dimensional integrals and necessitate solving only the photon frequency integral along the ray, as is done in our reverse ray-tracing formulation (Younsi, Wu & Fuerst 2012, submitted). This could be performed with a Gauss-Legendre quadrature with a carefully selected number of points.

We now present a method to evaluate the λ-integral. Since Q_n is independent of x (and therefore λ), we need not worry about this term in $\mathcal{M}_n$. However, we must evaluate the integral of B_n over λ. Using equation (82) and integrating by parts it is easily verified that

$$ \int d\lambda B_n e^{-\lambda/\tau} = -(\gamma^{-1}+\gamma'^{-1})e^{-\lambda/\tau}S_{n,2} - \int d\lambda e^{-\lambda/\tau}(R_n+R_{n+1}) + \int d\lambda e^{-\lambda/\tau}\left[-\gamma\gamma' S_{n,0} + 2S_{n,1} + \left(\frac{2}{\gamma\gamma'} - \frac{1}{\tau}\right)S_{n,2}\right] . \tag{129} $$

All of these integrals are of the form

$$ T = \int d\lambda\, e^{-\lambda/\tau} {}_2\mathrm{F}_1\left[a, b; b+1; \frac{u}{2}(1-x^2)\right] . \tag{130} $$

It is natural to ask if it is possible to solve equation (130) analytically. In the $x = \gamma + \lambda$ case, i.e. before the scattering event, using this as the change of integration variable and expanding the exponential in a Taylor series, one finds

$$ T = e^{\gamma/\tau} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\tau^l} \int dx\, x^l\, {}_2\mathrm{F}_1\left[a, b; b+1; \frac{u}{2}(1-x^2)\right] . \tag{131} $$

We must now solve the integral in equation (131). We state the identity

$$ I_1 = \int dx\, x\ {}_2\mathrm{F}_1\left[a, b; b+1; \frac{u}{2}(1-x^2)\right] = -\frac{b}{(a-1)(b-1)u} {}_2\mathrm{F}_1\left[a-1, b-1; b; \frac{u}{2}(1-x^2)\right] . \tag{132} $$

Defining

$$I_p(a,b) = \int dx\, x^p \;\; {}_2\mathrm{F}_1[a,b;b+1;z] \ , \tag{133}$$

where $z \equiv u(1-x^2)/2$, it is easily shown by implementing equation (132) and integrating equation (131) by parts that one can define a recurrence relation

$$\frac{(a-1)(b-1)u}{b} I_p(a,b) = -x^{p-1}{}_2\mathrm{F}_1[a-1,b-1;b;z] + (p-1)I_{p-2}(a-1,b-1) \ . \tag{134}$$

The above relation requires we have seed values for I_0 and I_1. We have such a value for the latter, but not the former. To our knowledge, no closed form solution to I_0 exists. Thus we approach the problem via series expansion. As before $z \equiv \frac{u}{2}(1-x^2)$, which always satisfies $z \leqslant 1$ (recall $u = 1-\zeta$). There are four cases we must treat, namely $z = 0$, $z = 1$, $|z| < 1$ and $z \leqslant -1$:

$$I_0(z=0) \;\; = \;\; x, \tag{135}$$

$$I_0(z=1) \;\; = \;\; \frac{\Gamma(1-a)\Gamma(1+b)}{\Gamma(1-a+b)} x \ . \tag{136}$$

The latter two expressions are more involved. First, we detail $|z| < 1$:

$$\begin{aligned} I_0(|z|<1) &= \int dx {}_2F_1[a,b,b+1,z] \\ &= \int dx \sum_{n=0}^{\infty} f(n)(1-x^2)^n \\ &= x \sum_{n=0}^{\infty} f(n) {}_2F_1\left[\frac{1}{2}, -n, \frac{3}{2}, x^2\right] \ , \end{aligned} \tag{137}$$

where $f(n)$ is defined as

$$f(n) \equiv \frac{(a)_n (b)_n}{n!(b+1)_n} \left(\frac{u}{2}\right)^n \ . \tag{138}$$

For the final case we must consider analytic continuation of the Gauss hypergeometric function to the region $|z| \leqslant -1$. Consider the identity

$${}_2F_1[a,b,b+1,z] = (1-z)^{-a} {}_2F_1\left[a, 1, b+1, \frac{z}{z-1}\right] \ , \tag{139}$$

which maps the region $(-\infty, -1]$ to the region $(0,1)$. We may now consider a series expansion of the function as we are working in the correct domain. We found that it is best to formulate the problem this way. Note $z \equiv u(v-1)/2v$. Consider the substitution $v \equiv 1/x^2$ which implies $dx = -\frac{1}{2}v^{-3/2}dv$, hence the integral becomes, in the case $a = 3/2$:

$$\begin{aligned} I_0(z \leqslant -1) &= -\frac{1}{2} \int dv\, (1-z)^{-3/2} {}_2F_1\left[\frac{3}{2}, 1, b+1, \frac{z}{z-1}\right] v^{-3/2} \\ &= -\frac{1}{2} \sum_{n=0}^{\infty} g(n) \int dv\, (1-z)^{-3/2} z^n (z-1)^{-n} v^{-3/2} \\ &= -\frac{\sqrt{2}}{u^{3/2}} \sum_{n=0}^{\infty} g(n) \int dv\, (1-v)^n (1-\tilde{u}v)^{-(n+3/2)} \\ &= -\frac{\sqrt{2}}{u^{3/2}} \sum_{n=0}^{\infty} g(n) \int dv \left\{ \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} v^k \sum_{l=0}^{\infty} \frac{(n+3/2)_l}{l!} \tilde{u}^l v^l \right\} \\ &= -\frac{\sqrt{2}v}{u^{3/2}} \sum_{n=0}^{\infty} g(n) \left\{ \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-n)_k (n+3/2)_l}{k!l!(3/2+k+l-\frac{1}{2})} v^k (\tilde{u}v)^l \right\} \ , \end{aligned}$$

which is almost in the form required to be written as an Appell hypergeometric function. Employing the identity

$$\frac{1}{a+k+l-\frac{1}{2}} \equiv \frac{1}{a-\frac{1}{2}} \frac{(a-\frac{1}{2})_{k+l}}{(a+\frac{1}{2})_{k+l}} \ , \tag{140}$$

the final result becomes

$$I_0(z \leqslant -1) = -\frac{\sqrt{2}v}{u^{3/2}} \sum_{n=0}^{\infty} g(n) \mathrm{F}_1\left[1; -n, n+\frac{3}{2}; 2; v, \tilde{u}v\right] , \tag{141}$$

where $g(n)$ and $\tilde{u}$ are defined as

$$g(n) \;\; \equiv \;\; \frac{(a)_n (1)_n}{(b+1)_n n!} \ , \tag{142}$$

$$\tilde{u} \;\; \equiv \;\; \frac{u-2}{u} \ , \tag{143}$$

where in this instance $a = 3/2$ in $g(n)$. By a similar but slightly lengthier process we may derive the result for $a = 1/2$. We simply state the result:

$$I_0(z \leqslant -1) = \sqrt{\frac{2}{v(2-u)}} \sum_{n=0}^{\infty} g(n) \tilde{u}^n \mathrm{F}_1 \left[\frac{1}{2}; -n, n+\frac{1}{2}; \frac{3}{2}; \frac{1}{v}, \frac{1}{\tilde{u}v}\right] \ . \tag{144}$$

Due to the complexity of evaluating the Appell hypergeometric function, we wish to express equation (141) in a form which is convergent on the unit disc. Since this form applies for $z \leqslant -1$, it is easily shown that $|v| < 1$. However, $\tilde{u}v$ is not always in the region $[-1, 1]$. Consider the transformation

$$\mathrm{F}_1\left[\alpha; \beta_1, \beta_2; \gamma; z_1, z_2\right] = (1-z_1)^{\gamma-\alpha-\beta_1}(1-z_2)^{-\beta_2} \mathrm{F}_1 \left[\gamma-\alpha; \gamma-\beta_1-\beta_2, \beta_2, \gamma; z_1, \frac{z_2-z_1}{z_2-1}\right] \ . \tag{145}$$

Applying identity () to equation (141) we obtain the expression

$$I_0(z \leqslant -1) = -\frac{2^{a-1} v^{a-1/2}}{u^a (a-\frac{1}{2})} \sum_{n=0}^{\infty} g(n) \frac{(1-v)^{n+1}}{(1-\tilde{u}v)^{a+n}} \mathrm{F}_1 \left[1; \frac{1}{2}, a+n; a+\frac{1}{2}; v, \frac{1}{1-z}\right]. \tag{146}$$

Equation (146) is entirely convergent on the unit disc for all values of $z \leqslant -1$, thus implementation of the above formula should be simple with a truncated series expansion. We now need to find a convergent form or continuation of equation (144) . The following series expansion Burchnall & Chaundy (1940, 1941) is very useful, in particular for small x, y, as it converges rapidly

$$\mathrm{F}_1\left[\alpha; \beta_1, \beta_2; \gamma; x, y\right] = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta_1)_r (\beta_2)_r (\gamma-\alpha)_r}{r!(\gamma+r-1)_r (\gamma)_{2r}} (x\,y)^r {}_2\mathrm{F}_1\left[\alpha+r, \beta_1+r; \gamma+2r; x\right] \ {}_2\mathrm{F}_1\left[\alpha+r, \beta_2+r; \gamma+2r; y\right] \ . \tag{147}$$

Despite requiring two ${}_2\mathrm{F}_1$ function evaluations per series term we find it always converges in less than twenty terms for practically all convergent values of x and y, and often much less.

10 CONCLUSION

Here we present a derivation of a fully convariant Compton scattering kernel which are essential for formulating a self-consistent general relativistic radiative transfer in the presence of photon-electron scattering. Previous attempts to derive it were only partially successful, resulting in hybrid analytic-numericlal expressions, which limits their use in the explicit moment expansion of the integro-different form of the radiative transfer equation. We successfully obtain a close analytic expression of the scattering kernel in terms of hypergeometric functions. We show our derivations explicitly in full details. The expression that we obtain is numerical stable and the terms are well behaved in convergence. Such a close-form scattering kernel will greatly reduce the computation demands in solveing the general relativistic radiative transfer equation with Compton scattering.

REFERENCES

Abramowitz M., Stegun I. A., 1972, Handbook of Mathematical Functions

Appell P., 1880, Ann. Sci. Ecole Norm. Sup. 9 (2) (1880) 119-144, pp 119–144

Appell P., Kampé de Fériet J., 1926, Fonctions hypergomtriques et hypersphriques: polynomes d'Hermite. Paris: Gauthier-Villars, 1926

Baschek B., Efimov G. V., von Waldenfels W., Wehrse R., 1997, A&A, 317, 630

Bateman H., 1955, Higher transcendental functions

Beason J. D., Kershaw D. S., Prasad M. K., 1991, Journal of Computational Physics, 95, 497

Brinkmann W., 1984, J. Quant. Spec. Radiat. Transf., 31, 417

Burchnall J. L., Chaundy T. W., 1940, The Quarterly Journal of Mathematics, 11, 249

Burchnall J. L., Chaundy T. W., 1941, The Quarterly Journal of Mathematics, 12, 112

Colavecchia F. D., Gasaneo G., 2004, Computer Physics Communications, 157, 32

Colavecchia F. D., Gasaneo G., Miraglia J. E., 2001, Computer Physics Communications, 138, 29

Forrey R., 1997, Journal of Computational Physics, 137, 79

Fuerst S. V., 2006, PhD thesis, University College London

Fuerst S. V., Wu K., 2004, A&A, 424, 733

Fuerst S. V., Wu K., 2007, A&A, 474, 55

Jackson J. D., 1975, Classical Electrodynamics, 2nd ed., Wiley: New York

Kershaw D. S., Prasad M. K., Beason J. D., 1986, J. Quant. Spec. Radiat. Transf., 36, 273

Luke Y., 1969, The Special Functions and Their Approximations. No. v. 1 in Mathematics in Science and Engineering, Academic Press

Mihalas D., Mihalas B. W., 1984, Foundations of Radiation Hydrodynamics, Oxford University Press: Oxford

Nagirner D. I., Poutanen Y. J., 1993, Astronomy Letters, 19, 262
Olsson P. O. M., 1964, Journal of Mathematical Physics, 5, 420
Peraiah A., 2001, An Introduction to Radiative Transfer, Cambridge University Press: Cambridge
Poutanen J., Vurm I., 2010, ApJS, 189, 286
Prasad M. K., Kershaw D. S., Beason J. D., 1986, Applied Physics Letters, 48, 1193
Thorne K. S., 1981, MNRAS, 194, 439
Wu K., Fuerst S. V., Mizuno Y., Nishikawa K.-I., Branduardi-Raymont G., Lee K.-G., 2008, Chinese Journal of Astronomy and Astrophysics Supplement, 8, 226
Zhang S., Jin J., 1996, Computation of Special Functions, Wiley-Interscience