Cosmological equations for AGN luminosity functions

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1 K Correction

The redshift of extragalactic objects, such as AGN, makes it necessary to correct luminosities, so that all luminosities are over the same emitted band (without this correction all luminosities are over the same observed bands and hence different emitted bands for different redshifts). This emitted band is usually chosen to be the detection band, hence at $z = 0$ no K correction is applied.

$$L = L_{\text{obs}} \times K_{\text{corr}}(z)$$

where $L_{\text{obs}}$ is the luminosity in the observers’ bandpass and $L$ is the luminosity over the same bandpass in the emitting object’s frame of reference. For an object with a power law spectrum $F_\nu = k\nu^{-\alpha_0}$

$$K_{\text{corr}}(z) = (1 + z)^{\alpha_0 - 1}$$

This is the form of typical K corrections used in the X-ray regime.

The K correction is unity (and hence can be neglected) for $\alpha_0 = 1$. If $\alpha_0 \neq 1$ then neglecting the K-correction will result in spurious pure luminosity evolution with redshift.

In practice, not all AGN have the same spectral index, and the K corrections for individual AGN are not known. An average K correction is applied, equivalent to the assumption that all AGN have the same spectral index. In the appendix I discuss the more realistic assumption that AGN have a distribution of spectral slopes, and how this affects the AGN luminosity function and evolution.

2 Robertson-Walker metric

The Robertson-Walker metric encompasses all possible homogeneous, isotropic geometries. The metric is defined by:

$$ds^2 = -dt^2 = R^2(t)[d\eta^2 + \sin^2(\eta)d\zeta]$$

(1)

where

$$d\zeta = d\theta^2 + \sin^2(\theta)d\phi^2$$

and

$$\sin(x) = \sin(x) \quad \text{for ‘closed’ Universe (} k = +1 \text{)}$$

$$\sinh(x) = \sinh(x) \quad \text{for ‘open’ Universe (} k = -1 \text{)}$$

$$\sin(x) = (x) \quad \text{for ‘flat’ Universe (} k = 0 \text{)}$$

By substituting $r = \sin nx$ this can be rewritten as:

$$ds^2 = -dt^2 + R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\zeta \right]$$

(2)

The Hubble constant is defined by

$$H \equiv \left( \frac{\dot{R}}{R} \right)$$

and the subscript 0 denotes the present epoch, so that $H_0$ is the Hubble constant at the present time $t_0$. 

3 Friedmann Universe

A Friedmann model universe has been assumed (i.e. the universe is sufficiently rarefied that there is no pressure, and the cosmological constant $\Lambda$ is assumed to be zero.) Where possible, I have given equations which are applicable for any value of the cosmological deceleration parameter $q_0$. In this thesis I have used $q_0 = 0$ and $q_0 = 0.5$ which is common in the literature; equations relevant to both these specific cases are given here.

All observations have a limiting flux for detection, such that an object with a flux less than this limit cannot be distinguished from the background or noise of the detector. The flux over a band is related to the luminosity of a source by

$$S = \frac{L}{(4\pi D_l^2)K_{corr}}$$

where $S$ is flux, $L$ is intrinsic luminosity, and $K_{corr}$ is the K correction term. $D_l$ is the luminosity distance and is:

$$D_l = \frac{c(1 - q_0 + q_0 z + (q_0 - 1)(2q_0 z + 1)^{1/2})}{H_0 q_0^3}$$  \hspace{1cm} q_0 > 0

$$D_l = \frac{c(z + z^2/2)}{H_0} \hspace{1cm} q_0 = 0$$

$$D_l = \frac{2c((1 + z) - (1 + z)^{1/2})}{H_0} \hspace{1cm} q_0 = 0.5$$

where $c$ is the speed of light, $H_0$ is the Hubble constant, and $z$ is the redshift.

The limiting luminosity for detection varies with distance and so we are seeing different populations of AGN at different redshifts. To enable us to show what is observed in terms of numbers of objects at different distances and luminosities we use the luminosity function, $\phi$. This is defined as the number of objects detected per unit volume per unit luminosity interval, i.e.

$$\phi = \frac{d^2 N}{dV dL}$$

In general $\phi$ is a function of luminosity and redshift. For all calculations in this thesis, the comoving volume has been used rather than the observed volume. The comoving volume is the volume that a region of space would have if it was seen at the present cosmological epoch. This comoving volume is obtained by integrating the differential comoving volume, which is equal to $(1 + z)^3$ times the observed differential volume.

The differential observed volume is

$$dV_o = -cdtD_o^2 dA$$

where $V_o$ is the observed volume, $D_o$ is the angular distance, $t$ is light travel time and $A$ is solid angle.

$$D_o = D_l/(1 + z)^2$$

and

$$dt = \frac{-dz}{H_0(1 + z)^2} \hspace{1cm} q_0 = 0$$

$$dt = \frac{-dz}{H_0(1 + z)^2(1 + 2q_0 z)^{1/2}} \hspace{1cm} q_0 > 0$$

Hence the comoving volume, $V_c$ between here and redshift $z_1$ is:

$$V_c = A \int_0^{z_1} \frac{c^2(z + z^2/2)^2 dz}{H_0^4(1 + z)^3} \hspace{1cm} q_0 = 0$$

$$V_c = A \int_0^{z_1} \frac{c^2(q_0 z + (q_0 - 1)((1 + 2q_0 z)^{1/2} - 1))^2 dz}{H_0^4(1 + z)^3q_0^3(1 + 2q_0 z)^{1/2}} \hspace{1cm} q_0 > 0$$

There are analytical expressions for the volume in the special cases $q_0 = 0$ and $q_0 = 0.5$:

$$V_c = \frac{Ac^3}{8H_0^6}((1 + z)^2 - (1 + z)^{-2} - 4\ln(1 + z)) \hspace{1cm} q_0 = 0$$

$$V_c = \frac{8Ac^3}{H_0^6}((1 + z)^{-1} - (1 + z)^{-1/2} - 1/3(1 + z)^{-3/2} + 1/3) \hspace{1cm} q_0 = 0.5$$

The comoving volume is independent of the expansion of the Universe, and hence there is no density evolution of objects caused by this cosmological expansion.

Evolution in the comoving space density or luminosity of AGN, with redshift, is seen as a change in $\phi$. 

2
4 Friedmann-Lemaître Universe

A very useful text for this is Carroll & Press, 1992, Annu. Rev. Astron. Astrophys., 30, 499. For a non-zero cosmological constant $\Lambda$, we have energy density $\Omega_{\Lambda}$ as well as mass density $\Omega_M$, where

$$\Omega_{\Lambda} = \frac{\Lambda c^2}{3H_0^2} \quad (6)$$

The $c^2$ factor is often missed from the equation because units are chosen so that $c = 1$ (I think). It is useful to define

$$\kappa = 1 - \Omega_M - \Omega_{\Lambda}$$

The luminosity distance is then:

$$D_l = \frac{c(1+z)}{H_0 \sqrt{|\kappa|}} \times \sinh \left( \sqrt{|\kappa|} \int_0^z [(1+z')^2(1 + \Omega_M z') - z'(2+z')\Omega_{\Lambda}]^{-1/2} dz' \right) \quad \text{for } \kappa \neq 0 \quad (7)$$

where

$$\sinh(x) = \sin(x) \quad \text{for } \kappa < 0$$

$$\sinh(x) = \sinh(x) \quad \text{for } \kappa > 0$$

$$D_l = \frac{c(1+z)}{H_0} \times \left( \int_0^z [(1+z')^2(1 + \Omega_M z') - z'(2+z')\Omega_{\Lambda}]^{-1/2} dz' \right) \quad \text{for } \kappa = 0 \quad (8)$$

or more conveniently

$$D_l = \frac{c(1+z)}{H_0} \times \left( \int_0^z [\Omega_{\Lambda} + \Omega_M(1+z')]^{-1/2} dz' \right) \quad \text{for } \kappa = 0 \quad (9)$$


Now, to work out volumes, we return to the basic comoving differential volume element:

$$dV_c = -c dt D_m^2 (1+z)^3 dA$$

Note that

$$dt = \frac{-dz}{H_0(1+z)[(1+z)^2(1 + \Omega_M z) - z(2+z)\Omega_{\Lambda}]^{1/2}}$$

and remember that $D_m = D_l/(1+z)^2$. It is also useful to define the ‘proper motion distance’ $D_m = D_l/(1+z)$.

4.1 Curved space ($\kappa \neq 0$)

We will start with the comoving volume for closed space ($\kappa < 0$). If we differentiate $D_m$ with respect to $z$ we get:

$$\frac{dD_m}{dz} = \frac{c}{H_0} \times [(1+z')^2(1 + \Omega_M z') - z'(2+z')\Omega_{\Lambda}]^{-1/2} \times \cos \left( \sqrt{|\kappa|} \int_0^z [(1+z')^2(1 + \Omega_M z') - z'(2+z')\Omega_{\Lambda}]^{-1/2} dz' \right)$$

We now find that

$$dV_c = \frac{dA D_m^2}{\cos \left( \sqrt{|\kappa|} \int_0^z [(1+z')^2(1 + \Omega_M z') - z'(2+z')\Omega_{\Lambda}]^{-1/2} dz' \right)} \quad (10)$$

If we define the variable

$$X = \sqrt{|\kappa|} \int_0^z [(1+z')^2(1 + \Omega_M z') - z'(2+z')\Omega_{\Lambda}]^{-1/2} dz'$$

we have

$$D_m = \frac{c}{H_0 \sqrt{|\kappa|}} \sin(X)$$

and

$$dD_m = \frac{c}{H_0 \sqrt{|\kappa|}} \cos(X) dX$$
so that
\[ dV_c = \frac{c^3}{H_0^2(|\kappa|)^{3/2}} \sin^2 X dX \] (12)

Now, using the double angle formulae, we can use
\[ \sin^2(X) = \frac{1 - \cos(2X)}{2} \]
and we can then rewrite equation 12 as:
\[ dV_c = \frac{c^3}{2 H_0^2(|\kappa|)^{3/2}} [1 - \cos(2X)] dX \]

This can be integrated analytically to give us
\[ V_c = \frac{c^3}{4 H_0^2(|\kappa|)^{3/2}} [2X - \sin(2X)] \]

This expression will reduce to equation 27 of Carroll et al with the suitable substitution.

Now we consider open space (\( \kappa > 0 \)). This case is identical to the one we have just considered, except that we use hyperbolic functions instead of sine and cosine. Now instead of equation 12 we obtain
\[ dV_c = \frac{c^3}{H_0^2(|\kappa|)^{3/2}} \sinh^2 X dX \]

We use the hyperbolic double angle formula:
\[ \sinh^2(X) = \frac{\cosh(2X) - 1}{2} \]
to get
\[ dV_c = \frac{c^3}{2 H_0^2(|\kappa|)^{3/2}} [\cosh(2X) - 1] dX \]
and
\[ V_c = \frac{c^3}{4 H_0^2(|\kappa|)^{3/2}} [\sinh(2X) - 2X] \]

Now, since \( \kappa \) carries sign to distinguish the two cases, we can write a general expression for the volume:
\[ V_c = \frac{c^3}{4 H_0^2 \kappa \sqrt{|\kappa|}} [\sinh(2X) - 2X] \] (13)

4.2 Flat space (\( \kappa = 0 \))
This time we obtain a much simpler version of equation 11:
\[ dV_c = dA \ D_m^2 \ Dm \]
which leads to
\[ V_c = \frac{dA \ D_m^5}{3} \]

5 Pure Luminosity Evolution
PLE has been a successful model for the evolution of AGN at X–ray, optical, and radio wavelengths, although deviations from this model are seen at high redshift. Most evolutionary models found in the literature for the last ten years have been based on PLE.

In PLE, only the luminosities of AGN evolve, at a rate which is the same for AGN of all luminosities. The space number density of objects remains constant, which is equivalent to:
\[ \phi(L, z) dL = \phi_0(Lo) dLo \] (14)
or in integral form
\[ \int_{L_1}^{L_2} \phi(L, z) dL = \int_{L_{01}}^{L_{02}} \phi_0(L_0) dL_0 \]

where \( L_0 \) is the de-evolved (i.e. \( z = 0 \)) luminosity and \( \phi_0 \) is the de-evolved (i.e. \( z = 0 \)) luminosity function. This relation is used to determine the luminosity function at any redshift if the de-evolved \((z = 0)\) luminosity function is known. The evolution is independent of luminosity, hence depends only on redshift:
\[ L(z) = f(z) \times L_0 \]

Changing the variable in the right hand side of equation 14 to \( L \), we obtain
\[ \phi(L, z) = \frac{\phi_0(L/f(z))}{f(z)} \]

In the framework of a PLE model the XLF retains its shape at all redshifts, hence the XLF at any redshift depends on only the evolution law and the XLF at zero redshift (hereafter \( z = 0 \) XLF).

### 6 Pure Density Evolution

Although PDE does not fit the current observations very well, it is important to define it, since it may describe behaviour at high redshift where the luminosity function is less well understood, and good models of the luminosity function may be a combination of density evolution and luminosity evolution. For PDE:
\[ L(z) = L_0 \]
\[ \phi(z, L) = g(z) \phi_0(L) \]

where \( L_0 \) and \( \phi_0 \) are the luminosity and luminosity function respectively at \( z = 0 \). Due to the luminosity independence of \( g(z) \), the XLF retains its shape at all redshifts in PDE.

### 7 Other Observable Quantities Related to the Luminosity Function

It requires a large amount of observing time to obtain optical spectra of sufficient quality to give reliable redshifts, and hence distances, for faint samples of AGN. The number flux relation (number of sources \( N \) brighter than a flux \( S \) as a function of \( S \), often expressed as the log \( N - \log S \)) is used as a tool for studying evolution of faint samples without redshift information. The observed log \( N - \log S \) at faint fluxes can be compared to the predictions of evolution models, if the luminosity function is known from brighter samples. The number of objects per unit volume brighter than a given luminosity \( L \) is simply the integral of the differential luminosity function from \( L \) to infinity, and so the luminosity function is related to the number flux relation by:
\[ N(> S) = \int_0^\infty dz \int_{4\pi D_z^2 S}^{\infty} \phi(L, z) \frac{dV}{dz} dL \]

The luminosity function can also be used to predict the number of AGN seen as a function of redshift to a given limiting flux by:
\[ N(> S, z_1 < z < z_2) = \int_{z_1}^{z_2} dz \int_{4\pi D_z^2 S}^{\infty} \phi(L, z) \frac{dV}{dz} dL \]

The luminosity function is also an essential tool for calculating the fraction of the X-ray background due to the combined luminosity of AGN. This amounts to finding the X-ray flux per square degree. The flux of each object with \((L, z)\) multiplied by \( \phi(L, z) \) gives the flux per unit luminosity interval per unit volume. To obtain the total X-ray background we must integrate over volume and luminosity, but this time there is no limiting flux for detection, we simply integrate over given Luminosity (or de-evolved Luminosity) and redshift ranges.
\[ I_{XRB} = \int_0^{z_{\text{max}}} \int_{L_{\text{min}}}^{L_{\text{max}}} \phi(L, z) L \frac{dV}{dz} dL \]

\[ (4\pi D_z^2) K_{\text{corr}} \]